ZENITH / STUDY HUB

SPPU | Third Year Robotics & Automation | Semester V

PCC303-ROA • UNIT IV

DYNAMICS OF MANIPULATORS

Complete unit notes • 2024 Pattern

Theory • derivations • diagrams • numericals • exam answers

- Kinetic & potential energy
- Lagrange–Euler and Newton–Euler
- Dynamic equation & mass matrix
- Coriolis & centrifugal terms
- Motion planning and simulation examples

Original teaching guide: RKD Complete Exam Notes, Edition 3

Unit IV, source pages 74–92 | Compiled by Zenith

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How to use Unit IV

Course: PCC303-ROA | TE Robotics & Automation | Semester V

Official scope: SPPU 2024 Pattern, syllabus printed pages 19–20.

This unit contains all eight syllabus topics (08 teaching hours).

STEP 1 Read each In simple words explanation first.

STEP 2 Follow each energy / torque derivation line by line.

STEP 3 Cover the answer and recompute every numerical.

STEP 4 Redraw the free-body diagrams and algorithm flows.

STEP 5 Practise the Write this in the exam sections.

Included (source page numbers follow the existing Edition 3):

74–76 Energy and robot dynamics overview

77–78 Lagrange–Euler formulation and worked examples

79–81 Newton–Euler recursion and comparison

82–84 Equation of motion and complete 2R worked example

85–87 Mass matrix, Coriolis and centrifugal terms

88–89 Dynamic effects in motion planning

90–91 Simulation and revision questions

92 Unit question bank

Verification: symbolic 2R mass matrix and torque examples recomputed independently with SymPy. This edition is an extracted, unchanged section of the existing 121-page guide, not a newly peer-reviewed text.

Original diagrams were preserved from the source PDF.

Older papers are only supplementary practice, not a predicted format.

UNIT IV · 08 TEACHING HOURS

Unit IV · Dynamics of Manipulators IV

SYLLABUS TOPICS IN THIS UNIT

- **4.1** Kinetic & potential energy
- **4.2** Lagrange–Euler formulation
- **4.3** Newton–Euler formulation
- **4.4** Dynamic equation of motion
- **4.5** Mass matrix
- **4.6** Coriolis & centrifugal terms
- **4.7** Dynamic effects in motion planning
- **4.8** Simulation examples

AFTER THIS UNIT YOU CAN

Find the torques a robot's motors must give for a wanted motion, using energy (Lagrange) or forces (Newton–Euler), and explain each term of $M(q)\ddot{q} + C + G = \tau$ (CO4).

TYPICAL QUESTION TYPES (STUDY GUIDE, NOT AN OFFICIAL PATTERN)

Theory (6–7 marks): compare L–E and N–E, explain the terms of the dynamic equation, properties of the mass matrix, Coriolis vs centrifugal. Numericals (7 marks): torque of a one-link arm, equations of a 2-link arm, torque at a given state.

IN SIMPLE WORDS

Kinematics asks where and how fast. Dynamics asks how much torque the motors need to make that motion happen, against inertia and gravity.

THINK OF IT LIKE THIS

Swinging an empty bag is easy; swinging a full one is hard. And holding it still with your arm out (gravity) is tiring even with no motion. Dynamics puts numbers on both.

SYLLABUS TOPIC

4.1 Kinetic & Potential Energy

IN SIMPLE WORDS

Kinetic energy K is the energy of motion. Potential energy P is the energy of height (gravity). Lagrange's method builds the robot's equations from just these two.

Point mass: $K = \frac{1}{2}mv^2$ **Rotating body:** $K = \frac{1}{2}I\omega^2$ **Gravity:** $P = mgh$

Rigid link (general): $K = \frac{1}{2}mv_c^T v_c + \frac{1}{2}\omega^T I_c \omega$ (c = centre of mass)

For a whole robot: $K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, the sum of every link's K. The matrix $M(q)$ is the **mass matrix** (topic 4.5).

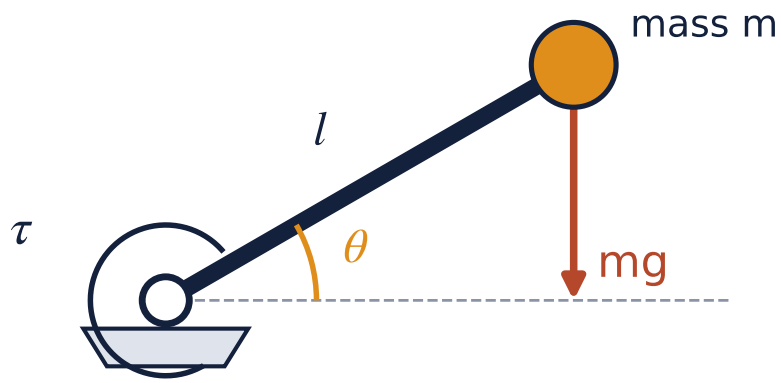


Fig. 23: One-link arm: a mass m at distance l, angle θ from the horizontal.

For this one-link arm the mass moves on a circle of radius l, so $v = l\dot{\theta}$ and its height is $h = l\sin\theta$:

$$K = \frac{1}{2}ml^2\dot{\theta}^2, \quad P = mgl\sin\theta$$

Solved example 4.1 · Energy of a one-link arm checked with sympy

Given: $m = 2$ kg, $l = 0.5$ m, $\theta = 30°$, $\dot{\theta} = 3$ rad/s, $g = 9.81$ m/s².

STEP 1 $K = \frac{1}{2} \times 2 \times 0.5^2 \times 3^2 = 2.250$ J

STEP 2 $P = 2 \times 9.81 \times 0.5 \times \sin 30° = 4.905$ J

ANSWER

$K = 2.250$ J (motion), $P = 4.905$ J (height).

SYLLABUS TOPIC

4.2 Lagrange–Euler Formulation

IN SIMPLE WORDS

Write the energies K and P, form $L = K - P$, and apply one standard formula. It gives the joint torques with no free-body diagrams.

$$L = K - P, \qquad \tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

choose joint variables $q_1, q_2, \ldots$

↓

position of each mass (FK) → its velocity

↓

kinetic energy K and potential energy P

↓

Lagrangian $L = K - P$

↓

for each joint: $\partial L/\partial \dot{q}_i$, its time derivative, and $\partial L/\partial q_i$

↓

$$\tau_i = \frac{d}{dt}(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i$$

Solved example 4.2 · Lagrange equation of a one-link arm

checked with sympy

Same arm: $K = \frac{1}{2}ml^2\dot{\theta}^2$, $P = mgl\sin\theta$.

STEP 1 $L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\sin\theta$

STEP 2 $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \;\rightarrow\; \frac{d}{dt}\left(ml^2\dot{\theta}\right) = ml^2\ddot{\theta}$

STEP 3 $\frac{\partial L}{\partial \theta} = -mgl\cos\theta$

STEP 4 Subtract:

$$\tau = ml^2\ddot{\theta} + mgl\cos\theta$$

STEP 5 **Numbers** ($m = 2$, $l = 0.5$, $\theta = 30°$, $\ddot{\theta} = 2$ rad/s²):

$$\tau = \underbrace{2(0.5)^2(2)}_{\text{inertia}} + \underbrace{2(9.81)(0.5)\cos 30°}_{\text{gravity}}$$

$$= 1.000 + 8.496 = 9.496 \text{ N·m}$$

ANSWER

$\tau = 9.496$ N·m. Most of it (8.496 N·m) just holds the arm up against gravity.

Solved example 4.3 · A sliding (prismatic) joint moving up

checked with sympy

Given: a 5 kg slide moves vertically with acceleration $\ddot{d} = 3$ m/s².

STEP 1 $K = \frac{1}{2}m\dot{d}^2$, $P = mgd$, so $L = \frac{1}{2}m\dot{d}^2 - mgd$.

STEP 2 $F = \frac{d}{dt}\left(m\dot{d}\right) - (-mg) = m\ddot{d} + mg$

STEP 3 $F = 5 \times 3 + 5 \times 9.81 = 15.00 + 49.05 = 64.05$ N

ANSWER

The actuator force is $F = 64.05$ N (for a prismatic joint, the "torque" is a force).

WRITE THIS IN THE EXAM

7 marks

Explain the Lagrange–Euler formulation and derive the equation of motion of a one-link manipulator.

L–E is an energy method: $L = K - P$ and $\tau_i = \frac{d}{dt}(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i$. **Steps:** joint variables → link velocities → K, P → L → apply the equation for each joint.

One-link arm (mass m at l, θ from horizontal): $K = \frac{1}{2}ml^2\dot{\theta}^2$, $P = mgl\sin\theta$, so $\tau = ml^2\ddot{\theta} + mgl\cos\theta$ (inertia + gravity). **Merits:** systematic, no internal forces, gives a closed-form model. **Demerit:** computation grows fast with DOF.

SYLLABUS TOPIC

4.3

Newton–Euler Formulation

IN SIMPLE WORDS

Newton–Euler uses forces directly ($F = ma$ and $\tau = I\alpha$) and works through the arm in two passes: velocities and accelerations go out, forces and torques come back.

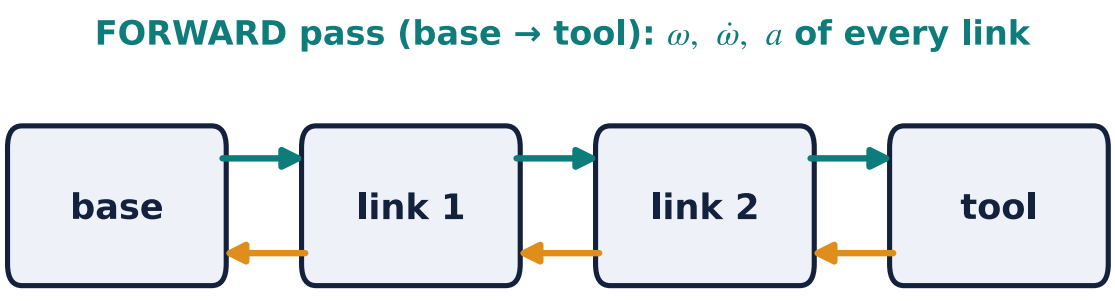


Fig. 24: The two passes of the recursive Newton–Euler algorithm.

1. **Forward pass (base → tool):** for each link find angular velocity ω, angular acceleration $\dot{\omega}$, and linear acceleration a of its centre of mass, using the joint motion.
2. **Each link:** force needed $F = ma_c$; moment needed $N = I\dot{\omega} + \omega \times I\omega$ (Euler's equation).
3. **Backward pass (tool → base):** each link must supply its own F, N **plus** whatever the next link pushes on it. Start at the tool (any payload force), move inward.
4. **Joint torque** = the component of the link's moment along its joint axis ($\tau_i = n_i \cdot \hat{z}_i$). For a prismatic joint, take the force component.

Solved example 4.4 · Newton–Euler for the one-link arm (same answer as Lagrange)

checked with sympy

Same arm: $m = 2$ kg, $l = 0.5$ m, $\theta = 30°$, $\dot{\theta} = 3$ rad/s, $\ddot{\theta} = 2$ rad/s².

STEP 1 **Forward pass:** the mass moves on a circle. Tangential acceleration $a_t = l\ddot{\theta} = 1.000$ m/s²; centripetal $a_n = l\dot{\theta}^2 = 4.500$ m/s² (towards the joint).

STEP 2 **Force on the mass:** $F = ma - mg$ (the link must also hold up the weight).

STEP 3 **Backward pass (torque about the joint):** only the tangential force and gravity's tangential part make torque; the centripetal pull passes through the joint and makes none:

$$\tau = l(ma_t) + l(mg\cos\theta)$$
$$= 0.5(2.000) + 0.5(16.991) = 9.496 \text{ N·m}$$

ANSWER

$\tau = 9.496$ N·m, **exactly the same** as Lagrange (checked by computer).

Lagrange–Euler vs Newton–Euler

	Lagrange–Euler	Newton–Euler
Based on	energy (scalar K, P)	forces and moments (vectors)
Method	differentiate $L = K - P$	recursive: forward pass, backward pass
Internal forces	not needed	found as a by-product
Output	closed-form M, C, G (good for analysis, design)	numbers for torques (good for real-time control)

COST | Symbolic L-E complexity depends on implementation.
Recursive N-E evaluates serial-chain inverse dynamics in O(n).

WRITE THIS IN THE EXAM — **7 marks**

Explain the Newton–Euler formulation. Compare it with Lagrange–Euler.

N–E applies Newton's law $F = ma_c$ and Euler's equation $N = I\dot{\omega} + \omega \times I\omega$ to each link, recursively. **Forward recursion** (base → tip): compute ω_i, $\dot{\omega}_i$, a_{ci} from joint motion. **Backward recursion** (tip → base): $f_i = f_{i+1} + F_i$, $n_i = n_{i+1} + N_i +$ moment terms; joint torque $\tau_i = n_i \cdot z_i$.

Comparison: L-E is energy-based and convenient for deriving symbolic robot models. Recursive N-E evaluates serial-chain inverse dynamics in O(n) operations. Both give the same torques when conventions and physical models match.

($O(n)$), so it is used for real-time control. Both give identical torques.

SYLLABUS TOPIC

4.4 Dynamic Equation of Motion

IN SIMPLE WORDS

Every robot, however complex, obeys one equation: torque = (mass × acceleration) + (speed effects) + (gravity).

$$\tau = M(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + G(q)$$

Term	Meaning
$M(q)\ddot{q}$	inertia: torque to **accelerate** the arm
$C(q,\dot{q})\dot{q}$	Coriolis + centrifugal: torque caused by joints **moving** (speed²)
$G(q)$	gravity: torque to **hold the arm up**, even at rest
τ	joint torques (forces for prismatic joints)

REMEMBER

Forward dynamics: given τ, find the motion: $\ddot{q} = M^{-1}(\tau - C\dot{q} - G)$ (used in simulation). **Inverse dynamics:** given the motion, find τ (used in control and motor sizing).

The 2-link arm, fully derived

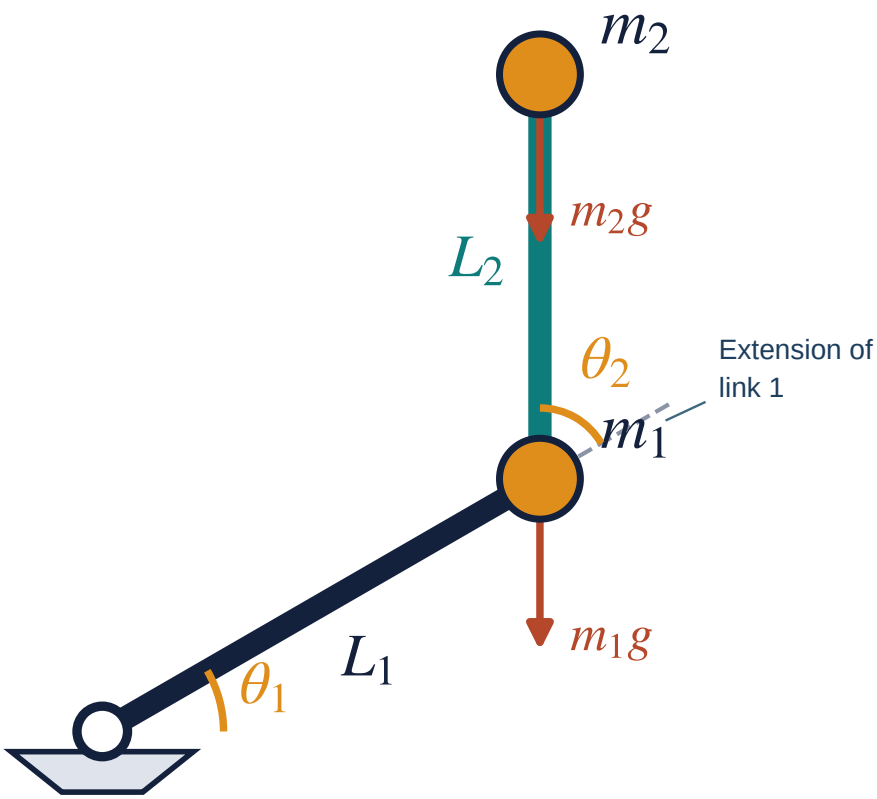


Fig. 25: point masses at link ends; theta1 from +x, theta2 relative to link 1.

Applying Lagrange to this arm (positions → velocities → K, P → L) gives:

$$\tau_1 = M_{11}\ddot{\theta}_1 + M_{12}\ddot{\theta}_2 - h\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) + G_1$$

$$\tau_2 = M_{12}\ddot{\theta}_1 + M_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 + G_2$$

$$M_{11} = (m_1 + m_2)L_1^2 + m_2 L_2^2 + 2m_2 L_1 L_2 c_2$$

$$M_{12} = m_2 L_2^2 + m_2 L_1 L_2 c_2, \quad M_{22} = m_2 L_2^2, \quad h = m_2 L_1 L_2 s_2$$

$$G_1 = (m_1 + m_2) g L_1 c_1 + m_2 g L_2 c_{12}, \quad G_2 = m_2 g L_2 c_{12}$$

checked with sympy

Solved example 4.5 · Torques of a 2-link arm at a given state

checked with sympy

Given: $m_1 = 2, m_2 = 1$ kg; $L_1 = 0.5, L_2 = 0.4$ m; $\theta = (30°, 60°)$; $\dot{\theta} = (1, 2)$ rad/s; $\ddot{\theta} = (1, 1)$ rad/s².

STEP 1 Mass matrix ($c_2 = \cos 60° = 0.5$):

$$M_{11} = 3(0.25) + 0.16 + 2(0.2)(0.5) = 1.11$$

$$M_{12} = 0.16 + 0.1 = 0.26, \quad M_{22} = 0.16$$

$$M = \begin{bmatrix} 1.1100 & 0.2600 \\ 0.2600 & 0.1600 \end{bmatrix}$$

STEP 2 Inertia part $M\ddot{\theta}$:

$$M\ddot{\theta} = \begin{bmatrix} 1.370 \\ 0.420 \end{bmatrix}$$

STEP 3 Coriolis / centrifugal ($h = 0.2 \sin 60° = 0.1732$):

$$C_1 = -h\left(2 \cdot 1 \cdot 2 + 2^2\right) = -1.386, \quad C_2 = h\left(1^2\right) = 0.173$$

STEP 4 Gravity ($\theta_1 + \theta_2 = 90°$, so $c_{12} = 0$):

$$G_1 = 3(9.81)(0.5)\cos 30° + 0 = 12.744, \quad G_2 = 0$$

STEP 5 Add up:

$$\tau_1 = 1.370 + (-1.386) + 12.744 = 12.728 \text{ N·m}$$

$$\tau_2 = 0.420 + 0.173 + 0.000 = 0.593 \text{ N·m}$$

ANSWER

$\tau = (12.728, 0.593)$ N·m. Gravity dominates joint 1; joint 2 is light because link 2 points straight up (no gravity torque).

WRITE THIS IN THE EXAM **7 marks**

Write the dynamic equation of a manipulator and explain each term. Derive it for a 2-link arm.

$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q)$. M: symmetric positive-definite inertia matrix (accelerations). $C\dot{q}$: Coriolis ($\dot{q}_i\dot{q}_j$) and centrifugal ($\dot{q}_i^2$) torques. G: gravity torques. τ: joint torques.

2R arm with point masses: write x_1, y_1, x_2, y_2; differentiate; $K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$; $P = m_1gy_1 + m_2gy_2$; apply Lagrange to get the two equations above with $M_{11}, M_{12}, M_{22}, h, G_1, G_2$.

SYLLABUS TOPIC

4.5 Mass Matrix

IN SIMPLE WORDS

The mass matrix $M(q)$ is the robot's "mass" in joint terms: how much torque each joint needs to accelerate. Unlike a single mass, it changes with the pose.

Properties (write all four):

1. **Symmetric:** $M_{ij} = M_{ji}$ (e.g. $M_{12} = M_{21}$).
2. Positive definite: $\dot{q}^T M(q)\,\dot{q} > 0$ for every nonzero $\dot{q}$, assuming independent coordinates and nondegenerate physical inertia.
3. **Depends on configuration:** for the 2R arm M_{11} contains $\cos\theta_2$: the arm is "heavier" to turn when stretched ($\theta_2 = 0$) than when folded.

4 **Diagonal terms** = effective inertia of each joint; **off-diagonal terms** = coupling (accelerating one joint pushes on another).

Solved example 4.6 · Check the mass-matrix properties

checked with sympy

From the previous example: $M = \begin{bmatrix} 1.1100 & 0.2600 \\ 0.2600 & 0.1600 \end{bmatrix}$.

STEP 1 Symmetric: $M_{12} = M_{21} = 0.26$ ✓

STEP 2 Positive definite: $M_{11} = 1.11 > 0$ and $\det M = 1.11(0.16) - 0.26^2 = 0.1776 - 0.0676 = 0.11 > 0$ ✓

STEP 3 Pose dependence: at $\theta_2 = 0$, $M_{11} = 0.75 + 0.16 + 0.4 = 1.31$; at $\theta_2 = 180°$, $M_{11} = 0.75 + 0.16 - 0.4 = 0.51$.

ANSWER

M is symmetric and positive definite, and joint 1 feels more than twice as heavy when the arm is stretched.

4.6 SYLLABUS TOPIC

Coriolis & Centrifugal Terms

IN SIMPLE WORDS

When joints move, extra torques appear just because of the motion. Centrifugal comes from one joint's speed squared; Coriolis comes from two joints moving together.

Term	Cause	Form
Centrifugal	a joint spinning pulls the outer links outward (like water in a spinning bucket)	$\dot{q}_i^2$
Coriolis	two joints moving at the same time; the moving link "drifts" sideways	$\dot{q}_i\dot{q}_j$

In the 2-link arm (with $h = m_2 L_1 L_2 \sin\theta_2$):

- $-2h\dot{\theta}_1\dot{\theta}_2$ in τ_1 → **Coriolis**.
- $-h\dot{\theta}_2^2$ in τ_1 and $+h\dot{\theta}_1^2$ in τ_2 → **centrifugal**.

REMEMBER

They grow with **speed squared**: tiny when moving slowly, important at high speed. They vanish when the arm is stopped, but gravity does not.

WRITE THIS IN THE EXAM 6 marks

Explain Coriolis and centrifugal forces in manipulator dynamics with an example.

They form the velocity-dependent term $C(q,\dot{q})\dot{q}$. **Centrifugal:** proportional to $\dot{q}_i^2$, from a single joint's rotation pulling outer masses outward. **Coriolis:** proportional to $\dot{q}_i\dot{q}_j$ $(i \neq j)$, from simultaneous motion of two joints. For a 2R arm: $\tau_1 \supset -m_2 L_1 L_2 s_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)$, $\tau_2 \supset m_2 L_1 L_2 s_2 \dot{\theta}_1^2$. They are significant at high speed and must be compensated in fast, accurate control.

SYLLABUS TOPIC

4.7 Dynamic Effects in Motion Planning

IN SIMPLE WORDS

A path that is fine slowly may need too much torque when done fast. Motion planning must respect the motors' torque limits.

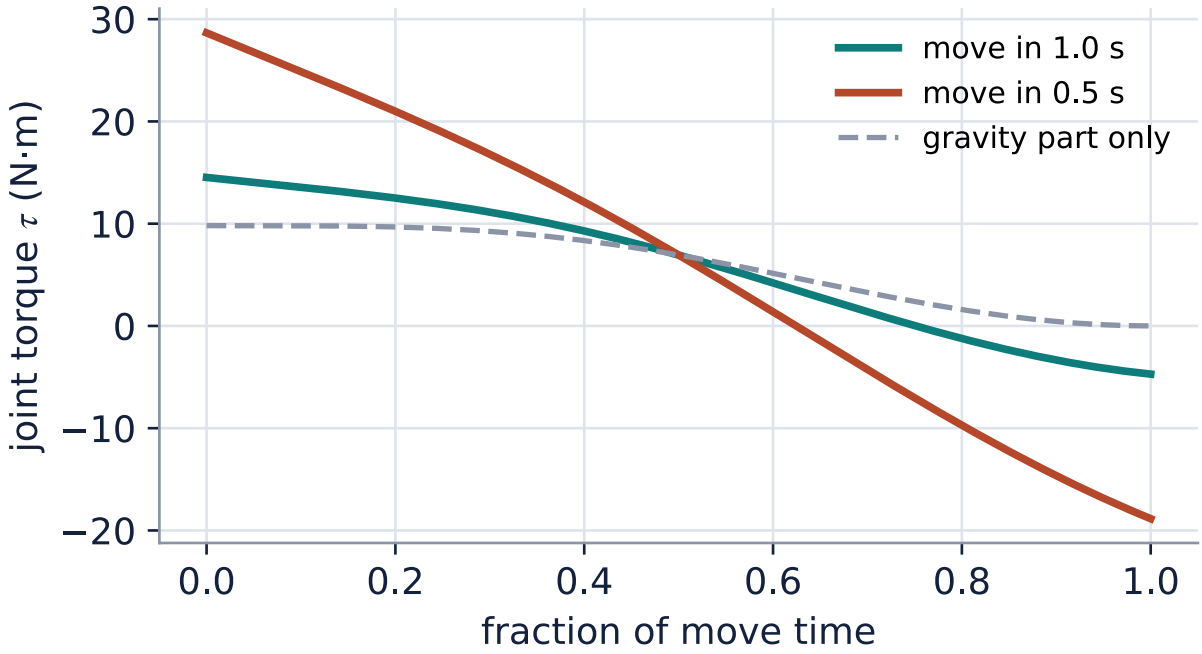


Fig. 26: Torque for a one-link arm lifting 0° → 90° along a smooth (cubic) path, done in 1 s and in 0.5 s.

What the graph shows:

- Moving slowly (1 s), the torque stays close to the **gravity part** (dashed).
- Moving twice as fast (0.5 s), the inertia part grows **4 times** (acceleration $\propto 1/t^2$), so the peak torque is much higher, and the motor may hit its limit.

So planners must:

1. keep $|\tau| \leq \tau_{\max}$ at every instant (torque limits);
2. use smooth trajectories (no jumps in acceleration) to avoid vibration (Unit V);

3. slow down near singularities or with heavy payloads;
4. use the dynamic model for **time-optimal** planning: as fast as possible without exceeding any limit.

WRITE THIS IN THE EXAM

6 marks

Explain the effect of dynamics on motion planning.

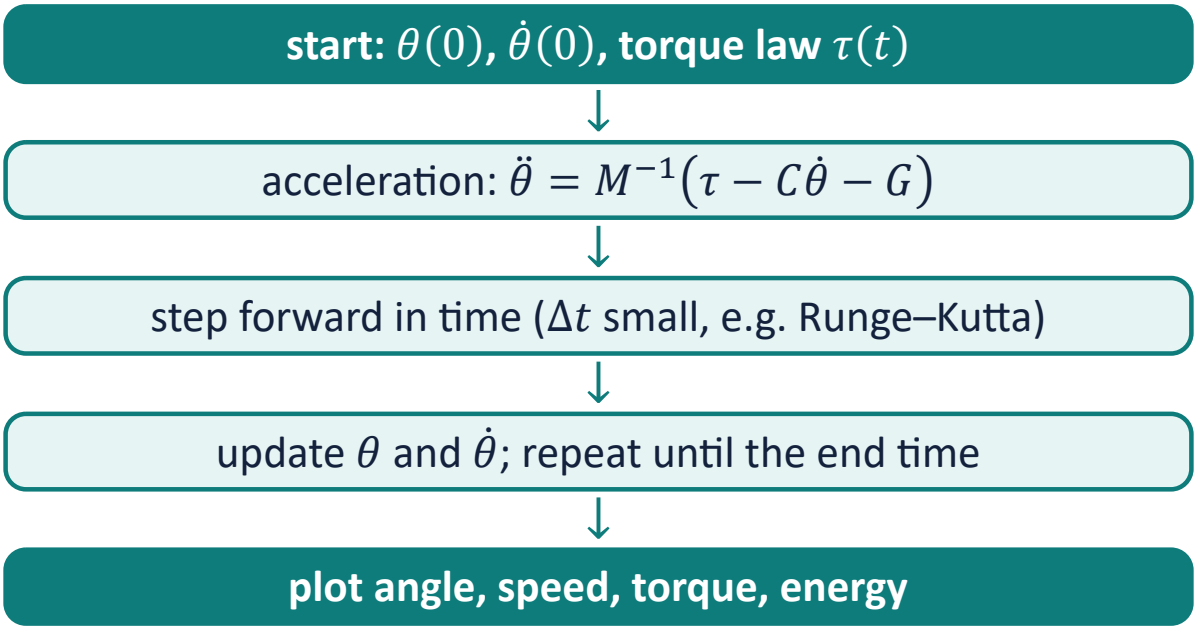
The torque needed is $\tau = M\ddot{q} + C\dot{q} + G$. Faster trajectories raise accelerations ($\propto 1/t^2$) and velocity terms ($\propto (1/t)^2$), so required torques can exceed actuator limits, causing tracking errors or saturation. Planners therefore (1) check torque / velocity / acceleration limits along the path, (2) use smooth (cubic, quintic) profiles to avoid jerk, (3) scale time to stay feasible (time-optimal planning), and (4) account for payload and gravity.

SYLLABUS TOPIC

4.8 Simulation Examples

IN SIMPLE WORDS

Simulation = solving the equation of motion on a computer, step by small step, to see how the robot will move before building or running it.



Solved example 4.7 · Simulate a one-link arm swinging freely

checked with sympy

Setup: $m = 2$ kg, $l = 0.5$ m, released at rest from the horizontal ($\theta = 0$), no motor torque ($\tau = 0$). Equation: $\ddot{\theta} = -(g/l)\cos\theta$. Solved with a 4th-order Runge–Kutta method, step $\Delta t = 0.002$ s, for 4.0 s.

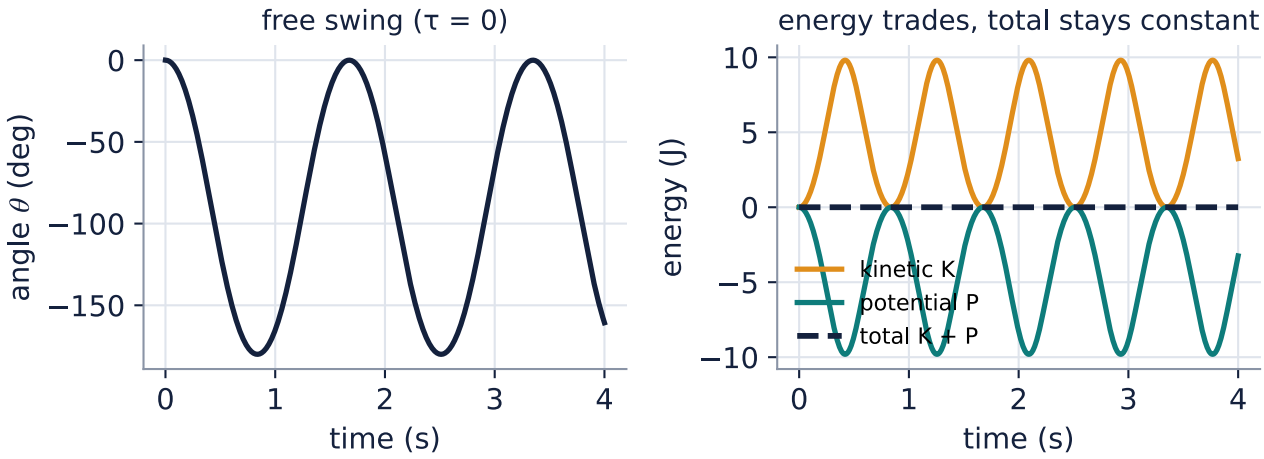


Fig. 27: Left: the arm swings down to −180° and back. Right: kinetic and potential energy trade places.

ANSWER | The ideal free arm swings between 0 and -180 degrees. Small energy drift supports consistency, but does NOT prove the simulation correct. Check equations, step-size convergence and reference solutions.

Tools used in practice: MATLAB / Simulink (Robotics System Toolbox), Python, ROS + Gazebo, CoppeliaSim, Webots.

Unit IV · Check yourself

1. Write the Lagrange equation for joint i.
2. Name the three terms of $\tau = M\ddot{q} + C\dot{q} + G$.
3. Which method is faster for real-time control, L–E or N–E?
4. State two properties of the mass matrix.
5. Coriolis term depends on ___; centrifugal on ___.
6. What is "forward dynamics" used for?
7. Why does moving twice as fast need much more torque?

ANSWERS

1 · $\tau_i = \frac{d}{dt}(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i$, $L = K - P$. 2 · Inertia, Coriolis/centrifugal, gravity. 3 · N–E. 4 · Symmetric, positive definite (also pose-dependent). 5 · $\dot{q}_i\dot{q}_j$; $\dot{q}_i^2$. 6 · Simulation (find motion from torque). 7 · Acceleration scales with $1/t^2$: 4× at half the time.

Unit IV · Question bank

No.	Question	M
1	Write expressions for kinetic and potential energy of a manipulator.	6
2	Explain the Lagrange–Euler formulation; derive the equation of a one-link arm.	7
3	Derive the dynamic equations of a 2R planar arm by Lagrange.	7
4	Explain the recursive Newton–Euler formulation (forward and backward).	7
5	Compare Lagrange–Euler and Newton–Euler methods.	6
6	Write the general dynamic equation and explain each term.	6
7	State and explain the properties of the mass (inertia) matrix.	6
8	Explain Coriolis and centrifugal terms with an example.	6
9	Explain the dynamic effects in motion planning.	6
10	Numerical: torque of a one-link arm / 2R arm at a given state.	7
11	Explain how a manipulator is simulated; forward vs inverse dynamics.	6