ZENITH STUDY HUB

SPPU TE Robotics & Automation, 2024 Pattern

Robot Kinematics & Dynamics

Unit III

Differential Motion & Velocity Kinematics

IN SIMPLE WORDS

Seven syllabus topics, each treated with the approach it needs: motion diagrams, clear derivations, solved numerical steps, practical precautions and exam practice. One official unit, organized for first-time learning and revision.

Official topic map

1. Linear & Angular Velocity
2. Jacobian Matrix Formulation
3. Relation between joint and end-effector velocities
4. Singularity Analysis
5. Manipulability and Condition Number
6. Velocity Mapping
7. Static Force Relationships

Source: official SPPU PCC303-ROA syllabus, printed p. 20. Technical reference: Modern Robotics Chapter 5. Original teaching diagrams and examples.

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Unit III | Topic navigator

RKD • Seven chapters • Click a topic to jump to its starting page

The combined source uses separate chapter page numbers in its original footers.
Use the unit-wide PDF page numbers below; the original chapters are preserved.

01 **Linear and Angular Velocity** — PDF page 3
Vectors, rotating links and instantaneous rates

02 **Jacobian Matrix Formulation** — PDF page 9
Analytical and geometric Jacobians

03 **Joint / End-Effector Velocities** — PDF page 12
Forward and inverse velocity mapping

04 **Singularity Analysis** — PDF page 15
Rank loss, determinant and damping

05 **Manipulability and Condition Number** — PDF page 18
Velocity ellipse and singular values

06 **Velocity Mapping** — PDF page 21
Control commands and constraints

07 **Static Force Relationships** — PDF page 24
Virtual work and joint torque

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Chapter content unchanged. This is a navigation improvement, not academic certification.

1 What does velocity mean for a robot?

TOPIC TYPE: Vector kinematics + diagrams + numericals **LEARNING GOAL: Relate motion of individual joints to motion of a point on the robot.**

IN SIMPLE WORDS

Position tells us *where* an object is. Velocity tells us how fast its position changes and in which direction. A robot's joint may be spinning, but the tool point usually follows a curved path. To plan its motion, we need both its linear velocity (metres per second) and its angular velocity (radians per second).

1.1 Three distinct ideas: position, velocity and acceleration

A point has position $\mathbf{r}(t) = [x(t), y(t), z(t)]^T$ relative to a chosen coordinate frame. Velocity is the first time derivative of position; acceleration is the second:

$$\boxed{\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}}, \qquad \boxed{\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}}.$$

A robot tool moving toward a workpiece can have a fixed magnitude of speed yet a changing velocity vector if its direction changes. Velocity is a vector; speed is its nonnegative magnitude $\|\mathbf{v}\|$. A fixed point on the robot base has zero velocity relative to the base, whereas a point on a rotating link generally does not.

Quantity	Meaning	Typical units
Position r	Location relative to chosen frame	m
Linear velocity v	Rate and direction of position change	m/s
Linear speed v	Magnitude $\|\mathbf{v}\|$	m/s
Angular velocity $\boldsymbol{\omega}$	Rate and axis of rotation, right-hand rule	rad/s
Linear acceleration a	Rate of change of velocity	m/s^2

1.2 Angular velocity: how rotation speed becomes a vector

For a planar rotating joint with angle $q(t)$, its scalar angular speed is $\dot{q} = dq/dt$. With counterclockwise positive viewed from the positive z direction, angular velocity is $\boldsymbol{\omega} = \dot{q}\,\hat{\mathbf{z}}$. For rotations about a general fixed axis $\hat{\mathbf{a}}$, $\boldsymbol{\omega} = \dot{q}\,\hat{\mathbf{a}}$. The axis direction follows the right-hand rule. An angular rate of 1 rad/s means the orientation angle increases by one radian per second, not one revolution per second.

EXAM TIP / COMMON MISTAKE

Radians are dimensionless mathematically, but writing rad/s prevents confusion with revolutions per minute. Convert $1\,\text{rev/min} = 2\pi/60\,\text{rad/s} \approx 0.10472\,\text{rad/s}$.

1.3 Velocity of a point on a rotating rigid link

Suppose point P is fixed on a rigid link a vector $\mathbf{r}_{P/O}$ from point O. If the link rotates with angular velocity $\boldsymbol{\omega}$ and O moves with velocity $\mathbf{v}_O$, then the point velocity is

$$\boxed{\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{P/O}}.$$

The cross product produces a direction perpendicular to both the rotation axis and the radius vector. In a plane with fixed pivot O, its magnitude is $v = |\omega| r$ when the radius is perpendicular to the axis. Velocity is tangent to the circular path, not outward along the link.

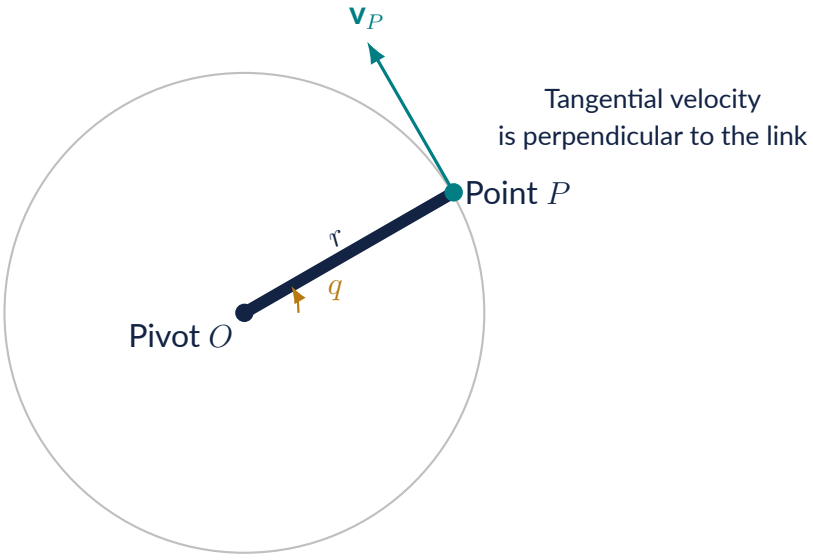

Figure. For counterclockwise rotation, the velocity of a link point is tangent to its circular path.

1.4 Derive the planar tangential velocity from calculus

For a point at radius L and angle $q(t)$, position is $x = L\cos q(t)$, $y = L\sin q(t)$. Apply the chain rule:

$$\boxed{\dot{x} = -L\sin q\,\dot{q}}, \qquad \boxed{\dot{y} = L\cos q\,\dot{q}}.$$

The magnitude is $\sqrt{\dot{x}^2 + \dot{y}^2} = L|\dot{q}|$ because $\sin^2 q + \cos^2 q = 1$. Geometrically, the velocity is obtained by turning the link direction by 90° and scaling it by angular speed. This provides a second check on the cross-product formula.

STEP-BY-STEP WORKED EXAMPLE

N01. Single rotating link. A link has length 0.50 m, angle 30° and angular rate $\dot{q} = 2\,\text{rad/s}$. Then

$$\dot{x} = -0.50\sin 30^\circ(2) = \boxed{-0.50\,\text{m/s}},$$

$$\dot{y} = 0.50\cos 30^\circ(2) = \boxed{+0.8660\,\text{m/s}}.$$

Speed is $\sqrt{(-.50)^2 + .8660^2} = 1.00$ m/s, equal to $L|\dot{q}| = 0.5 \times 2 = 1$ m/s. The negative x component means the tip is momentarily moving left as it rotates counterclockwise.

2 A two-joint robot has a tool velocity

IN SIMPLE WORDS

A moving robot tool is influenced by all upstream joints. Turning the shoulder moves both links. Turning the elbow moves the second link only. To calculate tool velocity, add the contribution from each joint rate.

2.1 Start from known position equations

For a planar two-revolute-joint arm,

$$x = L_1\cos q_1 + L_2\cos(q_1 + q_2),$$
$$y = L_1\sin q_1 + L_2\sin(q_1 + q_2).$$

Both angles vary with time. Differentiate each term using the chain rule:

$$\dot{x} = -L_1\sin q_1\,\dot{q}_1 - L_2\sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2),$$
$$\dot{y} = L_1\cos q_1\,\dot{q}_1 + L_2\cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2).$$

The quantity $\dot{q}_1 + \dot{q}_2$ occurs because link 2's global orientation is $q_1 + q_2$, and both joint rates affect it. The end-effector angular velocity in this planar chain is $\boldsymbol{\omega}_{\text{tool}} = (\dot{q}_1 + \dot{q}_2)\hat{\mathbf{z}}$, assuming its tool frame is rigidly aligned with link 2.

2.2 Gather coefficients into a Jacobian

The preceding two equations can be written in matrix form as $\dot{\mathbf{p}} = J(\mathbf{q})\dot{\mathbf{q}}$:

$$\boxed{J = \begin{bmatrix} -L_1\sin q_1 - L_2\sin(q_1 + q_2) & -L_2\sin(q_1 + q_2) \\ L_1\cos q_1 + L_2\cos(q_1 + q_2) & L_2\cos(q_1 + q_2) \end{bmatrix}}.$$

Each *column* answers a useful question: what direction and speed would the tool move if only this joint turned at one radian per second? This is the intuitive meaning of the Jacobian. A detailed derivation of its 3D form, singularities and condition number belongs to the remaining Unit III topics.

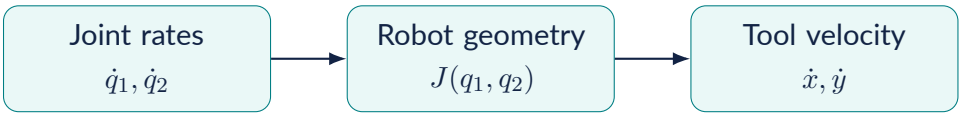

Figure. Instantaneous joint rates map to tool velocity through the configuration-dependent Jacobian.

STEP-BY-STEP WORKED EXAMPLE

N02. Moving planar 2R arm. Let $L_1 = 0.40$ m, $L_2 = 0.30$ m, $q_1 = 30°$, $q_2 = 60°$, $\dot{q}_1 = 1.0$ rad/s and $\dot{q}_2 = 0.5$ rad/s.

1. Check link orientations: $q_1 = 30°$, $q_1 + q_2 = 90°$.

2. Velocity components:

$$\dot{x} = -.4 \sin 30(1) - .3 \sin 90(1 + .5) = \boxed{-.650\,\text{m/s}},$$

$$\dot{y} = .4 \cos 30(1) + .3 \cos 90(1 + .5) = \boxed{+.3464\,\text{m/s}}.$$

3. Interpret: The tool is moving left and upward. Its speed is approximately $\sqrt{.650^2 + .3464^2} = 0.7366$ m/s. Its angular rate is $1 + .5 = 1.5$ rad/s about $+z$.

4. Independent check: Differentiate the FK equations symbolically or compute the Jacobian $J\dot{\mathbf{q}}$; it must produce the same two components.

2.3 Why tool speed can be zero when joints still turn

It is possible for two moving joints to produce tool velocities that cancel at an instant in a particular direction or task. This does not mean the joints are stationary; it means the effects of their motions sum to zero in that output coordinate. At singular configurations the Jacobian loses rank, so some tool motion directions cannot be generated instantaneously by any finite joint-speed combination. We will prove these conditions in the later Unit III topics.

EXAM TIP / COMMON MISTAKE

Do not confuse $\dot{q}_2$ with the absolute angular speed of link 2. The global orientation of link 2 is $q_1 + q_2$, so its angular speed is $\dot{q}_1 + \dot{q}_2$. Also keep joint speeds in rad/s and tool linear velocity in m/s.

3 Moving reference point and 3D vectors

3.1 The same formula works for a moving rigid link

If a rigid body has angular velocity $\boldsymbol{\omega}$ and point O on the body moves with velocity $\mathbf{v}_O$, the velocity of another point P is

$$\boxed{\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{P/O}}.$$

Choose and label *one common frame* for all three vectors before taking the cross product. The formula gives the velocity of a *point fixed in the rotating rigid body*; relative sliding motion requires an extra term. The angular velocity vector may be described with respect to space or body frames, but the components cannot be mixed without the appropriate rotation transform.

STEP-BY-STEP WORKED EXAMPLE

N03. Moving pivot and spinning link. Point O has $\mathbf{v}_O = (0.10, 0, 0)$ m/s. A link spins with $\boldsymbol{\omega} = (0, 0, 2)$ rad/s, and $\mathbf{r}_{P/O} = (0.20, -0.10, 0)$ m. Compute

$$\boldsymbol{\omega} \times \mathbf{r}_{P/O} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 2 \\ .2 & -.1 & 0 \end{vmatrix} = (.20, .40, 0)\,\text{m/s}.$$

Add the pivot velocity: $\boxed{\mathbf{v}_P = (0.30, 0.40, 0)\,\text{m/s}}$. Its speed is 0.50 m/s. The perpendicular cross-product term changes the tool's direction and magnitude.

3.2 Prismatic joints have direct linear speeds

A straight sliding joint measured by displacement $d(t)$ has joint rate $\dot{d}$ in m/s. If the slide axis is unit vector $\hat{\mathbf{a}}$, its velocity contribution to a point on the moving carriage is $\dot{d}\,\hat{\mathbf{a}}$ when the axis is fixed relative to the chosen frame. Example: a slide moving at 0.25 m/s along base $+x$ contributes $(0.25, 0, 0)$ m/s and has no rotation by itself. A rotating upstream frame can change the slide axis direction when expressed in the base frame, which is why complete robots need the Jacobian.

3.3 What a six-dimensional spatial tool velocity means

In spatial robotics, linear velocity is a 3-vector and angular velocity is also a 3-vector. They can be assembled into a six-component instantaneous motion representation (often a *twist*), but beware that the linear part of a spatial/body twist has a specific reference-frame and reference-point convention. It is not always identical to the plain derivative of the tool-frame origin expressed in another frame. For beginner numerical work, keep $\mathbf{v}_{\text{tool}}$ and $\boldsymbol{\omega}_{\text{tool}}$ separate until the frame convention is explicitly defined.

WRITE THIS IN THE EXAM

Model answer: Explain linear and angular velocity of a robot manipulator. Define $\mathbf{v} = d\mathbf{r}/dt$, speed $\|\mathbf{v}\|$, and $\boldsymbol{\omega} = \dot{q}\hat{\mathbf{a}}$. Draw a rotating rigid link with a tangential point velocity. Derive the planar formulas $\dot{x} = -L\sin q\dot{q}$, $\dot{y} = L\cos q\dot{q}$ using the chain rule and show $v = L|\dot{q}|$. State the rigid-body equation $\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{P/O}$, explain units and the chosen frame, then connect multiple joint rates to the tool through $\dot{\mathbf{p}} = J(\mathbf{q})\dot{\mathbf{q}}$ with a labeled example.

4 End-of-topic revision and practice

QUICK RECALL

Core formulas: $\mathbf{v} = d\mathbf{r}/dt$ (m/s); $\boldsymbol{\omega} = \dot{q}\hat{\mathbf{a}}$ (rad/s); $\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{P/O}$; $\dot{x} = -L \sin q\dot{q}$; $\dot{y} = L \cos q\dot{q}$; $v = L|\dot{q}|$ for a fixed pivot in plane; $\dot{\mathbf{p}} = J(\mathbf{q})\dot{\mathbf{q}}$.

4.1 Original self-test question bank

1. Define linear velocity, speed and angular velocity, giving SI units and an example of each.
2. Draw a revolute link and show why the tip velocity is perpendicular to the link.
3. Derive $\dot{x}$ and $\dot{y}$ of a 1R link using the chain rule.
4. For a 0.30 m link moving at 3 rad/s about a fixed pivot, determine tip speed. *Answer: 0.90 m/s.*
5. A robot joint turns at 30 rpm. Express the angular speed in rad/s. *Answer: π rad/s.*
6. Explain why $\dot{q}_1 + \dot{q}_2$ appears in the 2R tool angular velocity.
7. Differentiate a 2R forward model to obtain the Jacobian matrix.
8. Given $\mathbf{v}_O$ and $\boldsymbol{\omega}, \mathbf{r}_{P/O}$, calculate $\mathbf{v}_P$ with a cross product.
9. Explain the effect of reversing the sign of a revolute joint's rate on tip velocity direction.
10. Why must velocity vectors be expressed in a common frame before addition?

4.2 Source and transition to the next topic

Official topic: *Linear & Angular Velocity*, RKD Unit III, SPPU TE Robotics & Automation 2024 Pattern, printed syllabus p. 19 (original syllabus). Technical reference: Lynch and Park, Modern Robotics Chapter 5: Velocity Kinematics and Statics, and original Zenith calculations/figures. This compiled Unit III guide includes all seven official topic chapters. This section covers Topic 1 (Linear & Angular Velocity); Chapters 2 through 7 follow in the same PDF. No earlier-pattern examination question is attributed without a verified original source match.

1 A Jacobian connects joint motion to tool motion

TOPIC TYPE: Differentiation + matrices + geometric diagram LEARNING GOAL: Derive each Jacobian column, not memorize a matrix.

IN SIMPLE WORDS

A robot arm may have two, three or six joints. At a particular posture, moving any one joint a little makes the tool move in a particular direction. The **Jacobian matrix** collects those directions in columns. Multiplying by the joint speeds predicts the tool's instantaneous velocity.

1.1 Begin with forward kinematics

Let the planar 2R arm have lengths L_1, L_2 measured in metres and revolute angles q_1, q_2 measured in radians. With the base frame fixed and counterclockwise angles positive,

$$x = L_1 \cos q_1 + L_2 \cos(q_1 + q_2), \qquad y = L_1 \sin q_1 + L_2 \sin(q_1 + q_2).$$

Each of these expressions gives a position. To get velocity, differentiate with respect to time. The *chain rule* says $\frac{d}{dt} \sin q(t) = \cos q(t)\dot{q}$ and $\frac{d}{dt} \cos q(t) = -\sin q(t)\dot{q}$.

$$\dot{x} = \big[-L_1 \sin q_1 - L_2 \sin(q_1 + q_2)\big]\dot{q}_1 - L_2 \sin(q_1 + q_2)\dot{q}_2,$$
$$\dot{y} = \big[L_1 \cos q_1 + L_2 \cos(q_1 + q_2)\big]\dot{q}_1 + L_2 \cos(q_1 + q_2)\dot{q}_2.$$

Gather coefficients of $\dot{q}_1$ and $\dot{q}_2$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1 \sin q_1 - L_2 \sin(q_1 + q_2) & -L_2 \sin(q_1 + q_2) \\ L_1 \cos q_1 + L_2 \cos(q_1 + q_2) & L_2 \cos(q_1 + q_2) \end{bmatrix}}_{J(q_1, q_2)} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}.$$

1.2 Why the partial-derivative formula works

Write $\mathbf{p} = f(\mathbf{q})$, where $\mathbf{p} = [x, y]^T$ and $\mathbf{q} = [q_1, q_2]^T$. Then $\dot{\mathbf{p}} = \frac{\partial f}{\partial \mathbf{q}}\dot{\mathbf{q}}$, so $\boxed{J_{ij} = \frac{\partial p_i}{\partial q_j}}$. Column 1 says: move only the shoulder at unit angular speed; column 2 says: move only the elbow. They are measured in metres per radian for this planar position Jacobian.

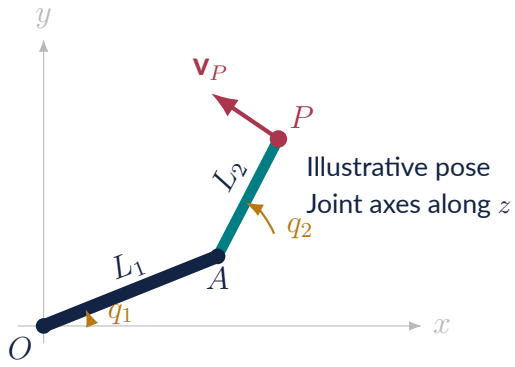

Figure. Original planar 2R schematic. Both joints rotate about the axis perpendicular to the page; the red arrow represents one possible tool-velocity direction.

1.3 Geometric construction without differentiating every time

For revolute joint i with axis $\mathbf{z}_i$ and a reference point $\mathbf{o}_i$ on that axis, the tool-origin linear-velocity column is $\mathbf{z}_i \times (\mathbf{p} - \mathbf{o}_i)$. Its angular-velocity column is $\mathbf{z}_i$. For a prismatic joint, the linear column is $\mathbf{z}_i$ and the angular column is zero. Express all vectors in the *same coordinate frame* before taking cross products.

Joint type	Tool linear Jacobian column	Tool angular Jacobian column
Revolute	$\mathbf{z}_i \times (\mathbf{p} - \mathbf{o}_i)$	$\mathbf{z}_i$
Prismatic	$\mathbf{z}_i$	$\mathbf{0}$

For a six-dimensional *geometric* tool velocity $[\mathbf{v}_P; \boldsymbol{\omega}]$, stacking these columns builds a $6 \times n$ geometric Jacobian. A body or space *twist Jacobian* uses related but frame-dependent velocity coordinates; do not silently equate the two without specifying their reference point and frame.

STEP-BY-STEP WORKED EXAMPLE

N01. Evaluate J at a real posture. Given $L_1 = 0.40$ m, $L_2 = 0.30$ m, $q_1 = 30°$, $q_2 = 60°$. First, $q_1 + q_2 = 90°$. Use $\sin 30 = 0.5$, $\sin 90 = 1$, $\cos 30 = 0.866025$, $\cos 90 = 0$:

$$J = \begin{bmatrix} -.4(.5) - .3(1) & -.3(1) \\ .4(.866025) + .3(0) & .3(0) \end{bmatrix} = \boxed{\begin{bmatrix} -.5000 & -.3000 \\ .3464 & 0 \end{bmatrix} \text{ m/rad}}.$$

If $\dot{\mathbf{q}} = [1, .5]^T$ rad/s, then $\dot{x} = -.5 - .15 = -.650$ m/s and $\dot{y} = .3464$ m/s. Check: both velocity entries have units of m/s.

STEP-BY-STEP WORKED EXAMPLE

N02. Derive Jacobian from a 1R link. For $\mathbf{p} = [L\cos q, L\sin q]^T$, differentiation gives

$$J(q) = \frac{\partial \mathbf{p}}{\partial q} = [-L\sin q,\ L\cos q]^T.$$

At $L = .5$ m, $q = 0$, $\dot{q} = 2$ rad/s, $J = [0, .5]^T$ m/rad. Therefore $\dot{\mathbf{p}} = [0, 1]^T$ m/s: the tip is instantaneously moving upward. The one-column Jacobian cannot independently command two arbitrary tool-velocity components.

EXAM TIP / COMMON MISTAKE

The 2R Jacobian above is a **2 by 2 position Jacobian**; a full robot end-effector may require a $6 \times n$ Jacobian. Count output velocity components and joint variables before deciding its size.

WRITE THIS IN THE EXAM

Derive the Jacobian for a planar 2R manipulator. (1) Sketch and label L_1, L_2, q_1, q_2. (2) Write x and y. (3) Differentiate using chain rule. (4) Collect coefficients in J. (5) State $\dot{\mathbf{p}} = J\dot{\mathbf{q}}$, units, and the common base frame.

QUICK RECALL

$J = \partial f / \partial \mathbf{q}$; revolute linear column $\mathbf{z} \times (\mathbf{p} - \mathbf{o})$; prismatic linear column $\mathbf{z}$; $\dot{\mathbf{p}} = J\dot{\mathbf{q}}$.

2 Practice

1. Draw and label the 2R Jacobian geometry.
2. Differentiate 1R position for an angle of $45°$.
3. Explain why the second link angle is $q_1 + q_2$.
4. State the size and units of the 2R Jacobian.
5. Construct revolute and prismatic geometric columns.
6. What changes when the frame changes?
7. Evaluate J at $q_1 = 0, q_2 = 90°$, $L_1 = L_2 = 1$ m.

Sources and conventions

SOURCE AND SCOPE

Official scope: SPPU PCC303-ROA, TE (2024 Pattern), Unit III, printed syllabus p. 20 (PDF). Technical reference: Lynch and Park, Modern Robotics, Chapter 5. All figures and worked examples here are original Zenith teaching examples, not reproduced textbook figures or verified past-paper questions. Planar 2R rigid-link assumptions and stated joint/frame conventions apply.

1 From motor rates to tool speed

TOPIC TYPE: Matrix multiplication + inverse solutions LEARNING GOAL: Calculate both forward and reverse instantaneous velocity problems.

IN SIMPLE WORDS

Joint encoders tell us how quickly motors are turning, but a welding tip or gripper needs a *Cartesian* velocity. The Jacobian translates between these descriptions at the present robot posture.

1.1 The forward velocity question

When $\mathbf{q}$ and $\dot{\mathbf{q}}$ are known, compute $\boxed{\dot{\mathbf{p}} = J(\mathbf{q})\dot{\mathbf{q}}}$. It is only an *instantaneous* relationship; changing posture changes J. A planar 2R arm needs two joint rates to determine the tool's two in-plane velocity components. For a 3D manipulator, angular motion and linear motion generally need to be accounted for separately.

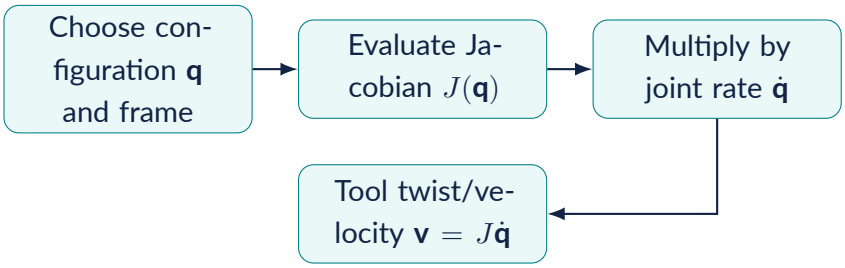

Figure. Original velocity-kinematics flow. The Jacobian must be recomputed when robot configuration changes.

STEP-BY-STEP WORKED EXAMPLE

N01. Forward velocity at a given posture. $L_1 = 0.4$ m, $L_2 = 0.3$ m, $(q_1, q_2) = (30°, 60°)$ and $(\dot{q}_1, \dot{q}_2) = (0.8, -0.2)$ rad/s. From the previous topic,

$$J = \begin{bmatrix} -.5 & -.3 \\ .346410 & 0 \end{bmatrix}, \quad \dot{\mathbf{p}} = \begin{bmatrix} -.5(.8) - .3(-.2) \\ .346410(.8) \end{bmatrix} = \boxed{\begin{bmatrix} -.3400 \\ .2771 \end{bmatrix} \text{ m/s}}.$$

The negative x component means the tool moves left even though the shoulder is rotating counterclockwise.

2 From requested tool speed back to joint speeds

For a *square, nonsingular* J, the unique instantaneous rate vector is $\dot{\mathbf{q}} = J^{-1}\dot{\mathbf{p}}$. For $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, provided $ad - bc \neq 0$,

$$J^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Do *not* use an ordinary inverse for rectangular matrices, and do not invert a singular matrix. For redundant or non-square cases, a pseudoinverse J^{+} can provide a least-norm or least-squares solution with caveats.

STEP-BY-STEP WORKED EXAMPLE

N02. Reverse velocity calculation. At $q_1 = 0$, $q_2 = 90°$, $L_1 = .40$ m, $L_2 = .30$ m:

$$J = \begin{bmatrix} -.3 & -.3 \\ .4 & 0 \end{bmatrix}.$$

The goal is $\dot{\mathbf{p}} = [.20, .12]^T$ m/s. The second row gives $.4\dot{q}_1 = .12$, so $\dot{q}_1 = \boxed{.30}$ rad/s. The first row gives $-.3(.3 + \dot{q}_2) = .20$, hence

$$\dot{q}_2 = -\frac{.20}{.30} - .30 = \boxed{-.9667 \text{ rad/s}}.$$

Check: $J[.3, -.9666667]^T = [.20, .12]^T$ m/s to rounding precision. The elbow must rotate clockwise in this example.

2.1 When there are more joints than required coordinates

Consider a simple redundant velocity model with two Cartesian components and three joint rates:

$$J = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \dot{\mathbf{p}} = [.2, .1]^T.$$

Infinitely many rate vectors satisfy $\dot{q}_1 + \dot{q}_3 = .2$ and $\dot{q}_2 + \dot{q}_3 = .1$. The minimum Euclidean-norm solution is $\dot{\mathbf{q}} = J^{+}\dot{\mathbf{p}} = [.1, 0, .1]^T$ (arbitrary consistent normalized units). In practical robot motion, joint limits and obstacles may require a different solution; the least-norm solution is not always the safest path.

2.2 Spatial six-component tool motion

For a specified reference frame and point, write $[\mathbf{v}_P; \boldsymbol{\omega}] = J_g(\mathbf{q})\dot{\mathbf{q}}$, where J_g is a geometric Jacobian. A six-axis industrial arm often has a 6×6 Jacobian, yet it may still be singular. Position-only and full-pose velocity goals are different tasks. Avoid mixing m/s with rad/s in numerical conditioning or weighting without an appropriate characteristic-length scale.

EXAM TIP / COMMON MISTAKE

An instantaneous inverse solution is not a full joint trajectory. After using a rate to update $\mathbf{q}$, recompute J; enforce joint-rate limits and avoid singularities.

WRITE THIS IN THE EXAM

Explain the relationship between joint and end-effector velocities. Start from forward kinematics $\mathbf{p} = f(\mathbf{q})$, use the chain rule $\dot{\mathbf{p}} = J(\mathbf{q})\dot{\mathbf{q}}$, state each symbol, draw the flow chart, and show a 2R multiplication. Explain the inverse only when J is square and nonsingular; mention pseudoinverse for other cases.

QUICK RECALL

Forward: $\dot{\mathbf{p}} = J\dot{\mathbf{q}}$. Inverse (square, nonsingular): $\dot{\mathbf{q}} = J^{-1}\dot{\mathbf{p}}$. Non-square: pseudoinverse or constrained optimization, with singularity checks.

3 Practice

1. Define instantaneous velocity mapping.
2. Explain why J depends on $\mathbf{q}$.
3. Solve $J[1, -2]^T$ for $J = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.
4. What does a pseudoinverse do?
5. Differentiate a scalar nonlinear $x(q)$ using the chain rule.
6. Describe why a velocity controller recomputes J each step.
7. Explain joint speed limit effects.

Sources and conventions

SOURCE AND SCOPE

Official scope: SPPU PCC303-ROA, TE (2024 Pattern), Unit III, printed syllabus p. 20 (PDF). Technical reference: Lynch and Park, Modern Robotics, Chapter 5. All figures and worked examples here are original Zenith teaching examples, not reproduced textbook figures or verified past-paper questions. Planar 2R rigid-link assumptions and stated joint/frame conventions apply.

1 When a robot loses an instantaneous direction

TOPIC TYPE: Rank + determinant + geometry LEARNING GOAL: Recognize singular configurations and understand why inverse rates explode nearby.

IN SIMPLE WORDS

A robot can reach a position yet temporarily be unable to move its tool in every desired direction. For example, when a two-link planar arm is fully straight, a small shoulder or elbow rotation initially moves the tip *perpendicular* to the links, not farther along their line. This is a kinematic singularity.

1.1 Derive the singularity test for a planar 2R arm

With the same J from Topic 02, expand $\det J$:

$$\begin{aligned}\det J &= \big(-L_1 \sin q_1 - L_2 \sin(q_1+q_2)\big) L_2 \cos(q_1+q_2) \\ &\quad - \big(-L_2 \sin(q_1+q_2)\big)\big(L_1 \cos q_1 + L_2 \cos(q_1+q_2)\big) \\ &= L_1 L_2 \sin\big((q_1+q_2)-q_1\big) = \boxed{L_1 L_2 \sin q_2}.\end{aligned}$$

Assuming nonzero link lengths, $\det J = 0$ when $q_2 = k\pi$, $k \in \mathbb{Z}$, so the links are fully straight or folded in this conventional 2R model. The *rank* drops below 2, meaning the two velocity columns become linearly dependent.

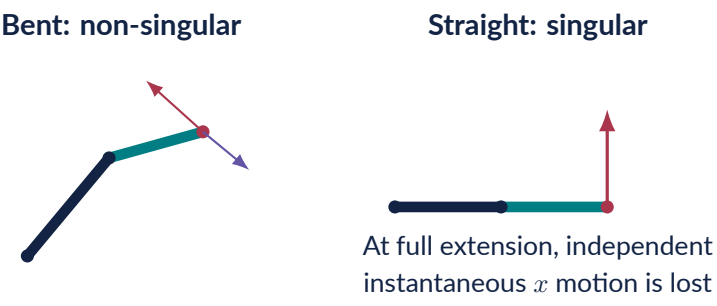

Figure. Two original teaching poses. The planar Jacobian loses rank at $q_2 = 0$ or $q_2 = \pi$ for nonzero link lengths.

STEP-BY-STEP WORKED EXAMPLE

N01. Detect an exact singularity. At $L_1 = L_2 = .5$ m and $q_1 = 0$, $q_2 = 0$,

$$J = \begin{bmatrix} 0 & 0 \\ 1 & .5 \end{bmatrix}, \qquad \det J = 0.$$

Any joint rates make $\dot{x} = 0$ at that instant. A commanded $+x$ tool velocity 0.1 m/s cannot be produced exactly by finite joint rates under the local linear model. Yet upward tool velocity $\dot{y} = \dot{q}_1 + .5\dot{q}_2$ is possible.

1.2 Near-singular is also dangerous

Singularity is not an on/off safety condition. When $|\det J|$ is small, a required inverse tool velocity can demand large joint speeds. Determinant magnitude depends on units and scaling, so modern implementations use singular values, the condition number and task-specific limits.

STEP-BY-STEP WORKED EXAMPLE

N02. Near extension. Let $L_1 = .4$, $L_2 = .3$ m and $q_2 = 5°$. Then

$$\det J = (.4)(.3)\sin 5° \approx \boxed{.010459\ \text{m}^2/\text{rad}^2},$$

versus $.12$ at $q_2 = 90°$. This is only 8.72% of the maximum absolute determinant at these link lengths. It warns that the posture is nearly singular; to judge controller risk, also inspect the singular values and requested motion direction.

1.3 The meaning of matrix rank

A 2×2 nonsingular Jacobian has two independent columns, so it can generate any *small* planar velocity direction. At rank 1, its achievable velocities lie along a single line. In 3D robots, the Jacobian has up to six spatial velocity components; singularity means rank is below the maximum achievable value for its task. A robot with fewer than six active degrees of freedom is intrinsically unable to command every six-dimensional twist; do not call every tall or short matrix "singular" without specifying the task rank.

1.4 What the controller should do

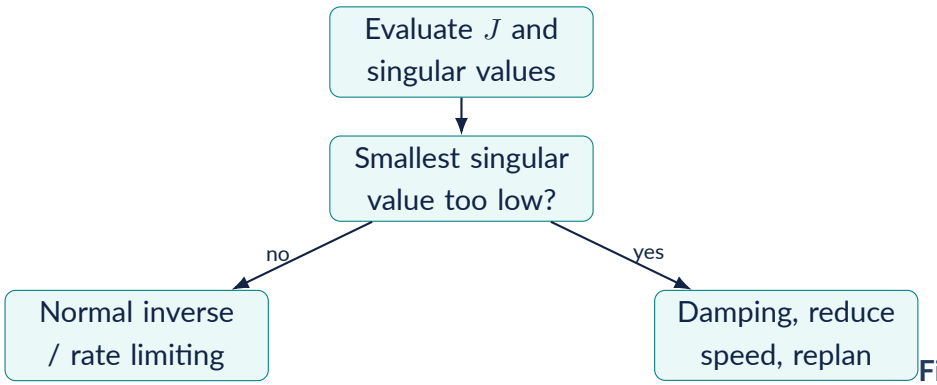

Figure. A conceptual safety flow; thresholds are controller-specific, not a universal syllabus constant.

One regularized least-squares rate law is

$$\boxed{\dot{\mathbf{q}} = J^T(JJ^T + \lambda^2 I)^{-1}\dot{\mathbf{p}}_{desired}}, \qquad \lambda > 0.$$

This damped least-squares expression avoids the infinite growth of an exact inverse near singularity but introduces task-velocity error. It is not a replacement for checking collision risk, joint limits and actuator capacities.

EXAM TIP / COMMON MISTAKE

At $q_2 = 180°$, the elbow is folded and also singular. Do not conclude that only fully stretched arms are singular.

WRITE THIS IN THE EXAM

Explain singularity with a diagram. Define Jacobian rank loss. Derive $\det J = L_1 L_2 \sin q_2$. Draw straight and bent postures, identify $q_2 = 0, \pi$, state the lost instantaneous velocity direction, and explain high inverse rates near singularity and regularization.

QUICK RECALL

2R singular when $L_1 L_2 \sin q_2 = 0$ for nonzero link lengths. Check rank or σ_{min}; near singularity use damping/speed limits and trajectory changes.

2 Practice

1. Define rank and kinematic singularity.
2. Derive the 2R determinant.
3. Test $q_2 = 0, 90, 180°$.
4. Show the lost tool direction for a straight arm.
5. Explain singular-value-based monitoring.
6. Why can a pseudoinverse still cause trouble near rank loss?
7. Describe one safe controller response.

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1 How easily can the robot move in each direction?

TOPIC TYPE: Singular values + velocity ellipse + numerical measures LEARNING GOAL: Explain shape, size and conditioning of the velocity set.

IN SIMPLE WORDS

Two robot postures can be equally far from a target but very different for velocity control. In one posture, small joint speeds allow tool motion in many directions. Near a straight-arm posture, most attainable motion points in nearly one direction. *Manipulability* is a family of measures describing this directional ability.

1.1 Construct the velocity ellipsoid

Assume joint-rate vectors satisfy $\|\dot{\mathbf{q}}\|_2 \leq 1$ in consistent normalized joint-rate units. Their images under J form a filled ellipse in 2D or an ellipsoid in higher dimensions. Singular value decomposition (SVD) writes

$$\boxed{J = U\Sigma V^T}, \quad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2), \quad \sigma_1 \geq \sigma_2 \geq 0.$$

The columns of U give the principal tool-velocity directions. The singular values σ_i give semiaxis lengths for a unit joint-rate sphere. If $\sigma_2 = 0$, the ellipse collapses to a line: the robot has a singularity.

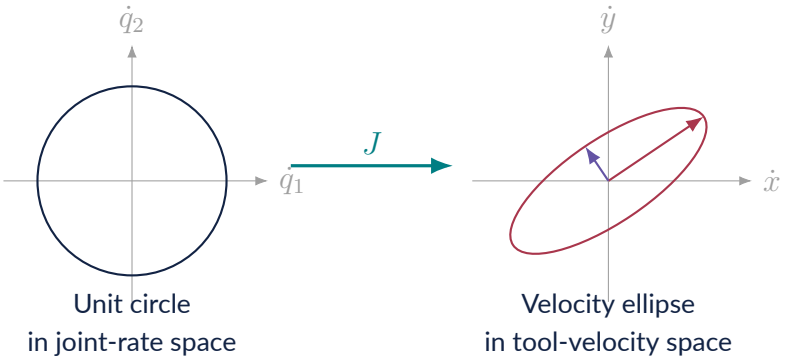

Figure. Original schematic: a unit joint-rate circle maps through J into an ellipse. Ellipse axis lengths are singular values (drawn schematically, not to numerical scale).

1.2 Three different numbers; do not confuse them

Measure	Formula	Interpretation
Minimum singular value	σ_{min}	Directly monitors weakest velocity direction
Condition number	$\kappa_2(J) = \sigma_{max}/\sigma_{min}$	Ratio of easiest to hardest directions; infinite at rank loss
Yoshikawa manipulability (full row rank)	$w = \sqrt{\det(JJ^T)}$	Product of nonzero output singular values; $\lvert\det J\rvert$ if square

Condition number depends on units and output-coordinate scaling. A six-dimensional Jacobian mixing metres/second and radians/second needs meaningful normalization before interpreting κ as a universal robot-quality score. w also has units derived from the selected Jacobian; never quote it as a unitless percentage.

STEP-BY-STEP WORKED EXAMPLE

N01. Compare shape and conditioning. For $L_1 = .4$ m, $L_2 = .3$ m, at $q_1 = 0$ and $q_2 = 90°$,

$$J = \begin{bmatrix} -.3 & -.3 \\ .4 & 0 \end{bmatrix}, \quad JJ^T = \begin{bmatrix} .18 & -.12 \\ -.12 & .16 \end{bmatrix}.$$

The eigenvalues of JJ^T are approximately 0.290417 and 0.049583 (units m^2/rad^2). Their square roots give $\sigma_1 \approx \boxed{.5389}$, $\sigma_2 \approx \boxed{.2227}$ m/rad, and

$$\boxed{\kappa_2 \approx 2.4201}, \qquad \boxed{w = |\det J| = .1200\ \text{m}^2/\text{rad}^2}.$$

The second direction is slower, but it is not lost; $\sigma_2 > 0$.

STEP-BY-STEP WORKED EXAMPLE

N02. Manipulability at two elbow angles. At fixed $L_1 = .4$ m, $L_2 = .3$ m the square 2R manipulability is $w = .12|\sin q_2|$ m^2/rad^2. Thus at $q_2 = 30°$, $w = .0600$; at $q_2 = 90°$, $w = .1200$; at $q_2 = 0$, $w = 0$. This conclusion is specific to the chosen 2R position Jacobian and this particular scalar measure.

1.3 Physical interpretation

When the ellipse is nearly circular, joint-rate capability is fairly balanced across tool directions. A long, thin ellipse means speed is easy in one direction and difficult in another. A collapsed ellipse indicates rank loss. Bigger manipulability is not always the same as safer or better performance: actuator limits, payload, workspace, collision constraints and accuracy can dominate.

EXAM TIP / COMMON MISTAKE

$\kappa(J)$ and $\kappa(JJ^T)$ are not equal: for a full-rank J, $\kappa_2(JJ^T) = \kappa_2(J)^2$. State which matrix is being conditioned.

WRITE THIS IN THE EXAM

Explain the manipulability ellipsoid and condition number. Define the image of unit joint rates under J. Sketch the circle-to-ellipse mapping. Explain U, Σ, V^T and principal axes; define w and κ_2. Show collapse at $\sigma_{min} = 0$ and note physical-unit normalization.

QUICK RECALL

$J = U\Sigma V^T$; ellipse semiaxes σ_i; $w = \sqrt{\det(JJ^T)}$ if full row rank; $\kappa_2 = \sigma_{max}/\sigma_{min}$ when $\sigma_{min} > 0$.

2 Practice

1. Describe a manipulability ellipse.
2. Define singular values.
3. Why does an ellipse collapse at singularity?
4. Calculate w for $L_1 = .25$, $L_2 = .50$, $q_2 = 90°$.
5. Compare $q_2 = 0$ and $90°$.
6. Explain the difference between $\kappa(J)$ and $\kappa(JJ^T)$.
7. Why do mixed units matter in a 6D Jacobian?

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1 Mapping velocities safely from joint space to task space

TOPIC TYPE: Forward mapping + controller steps + constraints **LEARNING GOAL: Use Jacobians to translate velocity commands in a practical robot loop.**

IN SIMPLE WORDS

Velocity mapping answers two questions: *If the motors move at known rates, which way does the tool move?* And *if the tool should move in a chosen direction, what motor rates should we command?* The forward question is always a Jacobian multiplication; the inverse question may need a pseudoinverse or a constrained controller.

1.1 Map in a common coordinate frame

For position-only planar motion, $\dot{\mathbf{p}} = J_p(\mathbf{q})\dot{\mathbf{q}}$. For a spatial rigid-body motion, state the reference *frame* and *point* explicitly. A geometric Jacobian maps joint rates to linear velocity of a specified point and angular velocity of the end effector, while a body/space twist uses its own coordinate convention. A 3D orientation's Euler angle rates are generally *not* the same as angular-velocity components; analytical and geometric Jacobians must not be interchanged without the conversion.

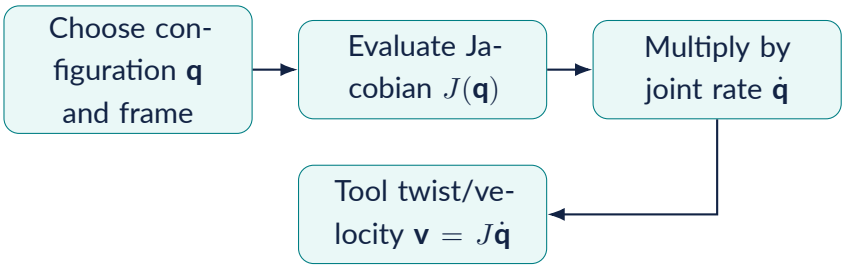

Figure. Original velocity-kinematics flow. The Jacobian must be recomputed when robot configuration changes.

1.2 Differential motion over a short time

For a small interval Δt, the first-order approximation is $\Delta\mathbf{p} \approx J(\mathbf{q})\Delta\mathbf{q}$, with $\Delta\mathbf{q} \approx \dot{\mathbf{q}}\Delta t$. This is accurate only to first order because J changes as the arm moves. For long motions, integrate while recomputing the current Jacobian and applying speed, acceleration and collision constraints.

STEP-BY-STEP WORKED EXAMPLE

N01. Forward mapping and tiny-step estimate. With $L_1 = .4$ m, $L_2 = .3$ m, $(q_1, q_2) = (30, 60)^\circ$ and $\dot{\mathbf{q}} = [1, .5]^T$ rad/s,

$$\dot{\mathbf{p}} = \begin{bmatrix} -.5 & -.3 \\ .346410 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ .5 \end{bmatrix} = \boxed{[-.650, .346410]^T \text{ m/s}}.$$

For $\Delta t = .02$ s, the first-order displacement is $\boxed{\Delta\mathbf{p} \approx [-.013, .006928]^T \text{ m}}$. These are approximate displacements, not exact new positions.

1.3 Turning a desired tool velocity into motor commands

For a full-rank square position Jacobian J, the inverse command $\dot{\mathbf{q}} = J^{-1}\dot{\mathbf{p}}_d$ is appropriate locally. In redundant arms, use $J^{+}\dot{\mathbf{p}}_d$ as a basic least-norm rate estimate; a null-space term $(I - J^{+}J)\mathbf{z}$ can tune joint behaviour without changing the task velocity for an exact full-row-rank mapping. At or near singularity, use damping and check residual velocity error.

STEP-BY-STEP WORKED EXAMPLE

N02. A motor command from a tip command. At $q_1 = 0$, $q_2 = 90^\circ$ and $(L_1, L_2) = (.4, .3)$ m, desire $\dot{\mathbf{p}}_d = [-.15, .08]^T$ m/s. The second Jacobian row gives $.4\dot{q}_1 = .08$, so $\dot{q}_1 = .20$ rad/s. The first row gives $-.3(\dot{q}_1 + \dot{q}_2) = -.15$, so $\dot{q}_2 = .30$ rad/s.

$$\boxed{\dot{\mathbf{q}}_d = [.20, .30]^T \text{ rad/s}}.$$

Check: $[-.3(.2 + .3), .4(.2)]^T = [-.15, .08]^T$ m/s.

1.4 A practical velocity-control pipeline

1. Measure joint positions $\mathbf{q}$ from encoders. Verify angle units and frame conventions.
2. Calculate FK and Jacobian $J(\mathbf{q})$ with the current tool geometry.
3. Specify desired tool velocity in the same frame as J.
4. Check singular values, limits and collisions; choose inverse, pseudoinverse or damped least-squares rate solver.
5. Clip or rescale to joint-rate limits and recalculate achieved task velocity $J\dot{\mathbf{q}}_{cmd}$.
6. Send rate commands, advance a short time, then re-measure and repeat.

EXAM TIP / COMMON MISTAKE

Do not promise the desired Cartesian velocity after clipping joint rates: recompute $J\dot{\mathbf{q}}_{cmd}$ to see what the actuators actually allow. Numerical velocity commands need safety monitors before use on a physical robot.

WRITE THIS IN THE EXAM

Explain velocity mapping of a manipulator. Define joint space and Cartesian task space, write $\dot{\mathbf{p}} = J\dot{\mathbf{q}}$, sketch the computation flow, demonstrate a 2R multiplication and inverse, and state what happens at singularity and when speed limits are applied.

QUICK RECALL

Forward: $\dot{\mathbf{p}} = J\dot{\mathbf{q}}$. Small step: $\Delta\mathbf{p} \approx J\dot{\mathbf{q}}\Delta t$. Inverse rate control: recompute J, check rank and limits, and verify achieved speed.

2 Practice

1. Define joint-space and task-space velocity.
2. Explain why frame consistency matters.
3. Calculate first-order displacement for a 0.1 s interval.
4. Explain Euler-angle rate versus angular velocity.
5. Write six safe velocity-loop steps.
6. Explain the null-space term in redundant control.
7. Describe the effect of clipping motor speed.

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1 Force at the robot tool becomes torque at its joints

TOPIC TYPE: Free-body diagram + power balance + transpose map LEARNING GOAL: Calculate static joint torques from a given end-effector force.

IN SIMPLE WORDS

Imagine pushing sideways at the tip of a robot arm. The force acts at the tool, but its load is felt as turning effort at the motor shafts. The same Jacobian that maps joint speeds to tool velocities can also map a tool force back to joint torques: use its *transpose*.

1.1 Derive the static mapping using virtual work

For a small permitted joint displacement $\delta\mathbf{q}$, the corresponding tool displacement is $\delta\mathbf{p} = J\delta\mathbf{q}$. Let $\mathbf{F}$ be an *external* force acting on the tool and $\boldsymbol{\tau}_{ext}$ the generalized torques it produces in the joint coordinates. Their virtual work is equal:

$$\delta W = \mathbf{F}^T\delta\mathbf{p} = \mathbf{F}^T J\delta\mathbf{q} = (J^T\mathbf{F})^T\delta\mathbf{q}.$$

Because the equality holds for arbitrary small $\delta\mathbf{q}$, it follows that

$$\boxed{\boldsymbol{\tau}_{ext} = J^T\mathbf{F}}.$$

The motor torques needed to hold the robot against that applied force are $-\boldsymbol{\tau}_{ext}$ *if* gravity, other loads and friction are excluded. Some books instead define $\mathbf{F}$ as the robot's output force, reversing signs in the equilibrium statement. Always name the force convention.

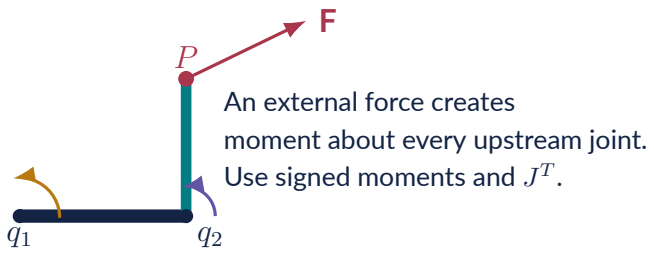

Figure. Original 2R force schematic; static torque signs depend on whether $\mathbf{F}$ is an external applied force or the robot's output force.

1.2 Check dimensions and why transpose matters

For a planar 2R position Jacobian, J is 2×2, force $\mathbf{F} = [F_x, F_y]^T$ is in newtons, and $J^T\mathbf{F}$ has units N m (torque). In a spatial robot, use a six-component wrench $\mathcal{F}$ (moment plus force with a clearly specified ordering and frame) and the corresponding six-dimensional twist Jacobian, yielding $\boldsymbol{\tau} = J^T\mathcal{F}$ in consistent dual coordinates. Different twist ordering conventions require permuting both velocity and wrench blocks consistently.

STEP-BY-STEP WORKED EXAMPLE

N01. Torque at a bent-arm posture. At $(q_1, q_2) = (0, 90^\circ)$, $(L_1, L_2) = (.4, .3)$ m,

$$J = \begin{bmatrix} -.3 & -.3 \\ .4 & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \text{ N}.$$

External joint moments are

$$\boldsymbol{\tau}_{ext} = J^T \mathbf{F} = \begin{bmatrix} -.3 & .4 \\ -.3 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ -3 \end{bmatrix} \text{ N m}}.$$

If no other loads act, motors holding this fixed pose must apply $[+1, +3]^T$ N m to oppose these external loads.

1.3 Power consistency proves the signs

Joint power from the external load is $\boldsymbol{\tau}_{ext}^T \dot{\mathbf{q}}$. Tool power is $\mathbf{F}^T \dot{\mathbf{p}}$. Because $\dot{\mathbf{p}} = J\dot{\mathbf{q}}$,

$$\boxed{\boldsymbol{\tau}_{ext}^T \dot{\mathbf{q}} = \mathbf{F}^T \dot{\mathbf{p}}},$$

provided all vectors use consistent sign, frame and reference-point conventions. This is an excellent numerical self-check.

STEP-BY-STEP WORKED EXAMPLE

N02. Verify the energy-rate identity. At $(q_1, q_2) = (30, 60)^\circ$, $(L_1, L_2) = (.4, .3)$ m, $\mathbf{F} = [10, 5]^T$ N and $\dot{\mathbf{q}} = [1, .5]^T$ rad/s,

$$J = \begin{bmatrix} -.5 & -.3 \\ .346410 & 0 \end{bmatrix}, \quad \boldsymbol{\tau}_{ext} = \begin{bmatrix} -5 + .346410(5) \\ -3 \end{bmatrix} = \begin{bmatrix} -3.267949 \\ -3 \end{bmatrix} \text{ N m}.$$

Joint power $= -3.267949(1) - 3(.5) = \boxed{-4.767949 \text{ W}}$. Tool velocity is $[-.65, .346410]^T$ m/s, so $\mathbf{F}^T \dot{\mathbf{p}} = 10(-.65) + 5(.346410) = \boxed{-4.767949 \text{ W}}$. The values match.

2 Limits of the static relation

This formula is a local virtual-work mapping, not a complete dynamic model. It ignores inertia and acceleration terms by itself. Gravity loads may need $\mathbf{g}(\mathbf{q})$, and robot friction, gear reduction and contact compliance can matter in real systems. With large external forces, singular postures and actuator torque limits require careful force-control design. *Static force relationships* do not imply a robot can safely hold any wrench simply because the matrix product can be computed.

EXAM TIP / COMMON MISTAKE

Never multiply $J\mathbf{F}$ here. Velocity maps forward via J, while conjugate force/torque maps backward via J^T. Also distinguish an external load's generalized torque from the opposing motor holding torque.

WRITE THIS IN THE EXAM

Derive the static force relation for a manipulator. Show $\delta\mathbf{p} = J\delta\mathbf{q}$, equate $\mathbf{F}^T\delta\mathbf{p}$ and $\boldsymbol{\tau}^T\delta\mathbf{q}$, conclude $\boldsymbol{\tau} = J^T\mathbf{F}$. Draw a free-body diagram, state force and sign conventions, and solve a 2R numeric example with N m units.

QUICK RECALL

Tool rate: $\dot{\mathbf{p}} = J\dot{\mathbf{q}}$. External-load generalized torque: $\boldsymbol{\tau}_{ext} = J^T\mathbf{F}$. Virtual power: $\tau_{ext}^T\dot{\mathbf{q}} = \mathbf{F}^T\dot{\mathbf{p}}$.

3 Practice

1. State the principle of virtual work.
2. Derive $J^T\mathbf{F}$.
3. Compute $J^T[4, 2]^T$ for $J = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
4. Explain external-force and motor-holding sign convention.
5. Prove the power identity.
6. Explain the 3D wrench extension.
7. State two dynamic effects omitted from this static formula.

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